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Couette flow of a smectic A liquid crystal

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Abstract

This paper considers the dynamics of cylindrically arranged parallel layers of smectic A liquid crystal subjected to Couette flow. Governing equations are constructed using a recently developed dynamic theory for smectic A (Stewart 2007 *Contin. Mech. Thermodyn.* **18** 343–60). These equations are solved to provide analytical solutions for the smectic layer undulations and velocity profiles. Results show the dependence of the response time of the smectic layers upon the permeation constant and the layer compression modulus. The relaxation times for the flow profiles are shown to depend upon two viscosities; estimates for these times are shown to be shorter than that for a typical approximation to the relaxation time of the smectic layer undulations.

1. Introduction

Liquid crystals are anisotropic fluids made up of elongated molecules which have an average molecular axis that aligns along a common direction in space which is usually denoted by the unit vector **n**, called the director. Smectic liquid crystals are layered structures with a well-defined interlayer distance. These layers may be described by a scalar function Φ, whereby the layer normal is given by $\mathbf{a} = \nabla\Phi/|\nabla\Phi|$. There are many different types of smectic liquid crystal although this paper shall deal only with smectic A (SmA). In most equilibrium situations, SmA liquid crystals form locally equidistant parallel layers in which the director **n** is parallel to the local unit layer normal **a** [2, 3]. However, in some equilibrium and non-equilibrium circumstances, it is believed that the director **n** and the layer normal **a** may decouple [1, 4–11].

As noted in [1], there are two ideas of compressibility in smectic liquid crystals: the first is the usual idea of compressibility in fluids and the second is the compressibility of the smectic layers themselves. We shall be dealing with an isothermal, incompressible fluid that possesses compressible smectic layers.

This paper considers the dynamics of a sample of SmA liquid crystal occupying the annular region $R_1 \leqslant r \leqslant R_2$ between two concentric cylinders in relative rotation and axial movement. The inner and outer cylinders have fixed radii R_1 and R_2 with corresponding azimuthal angular velocities V_1 and V_2, respectively, and corresponding axial velocities W_1 and W_2, respectively, as shown in figure 1. The usual cylindrical coordinate system (r, θ, z) having basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ will be used, with the z-axis coinciding with the common axis of the cylinders, i.e. perpendicular to the page. To highlight the main phenomena, we shall reduce the viscous stress tensor to include only the three main viscous coefficients

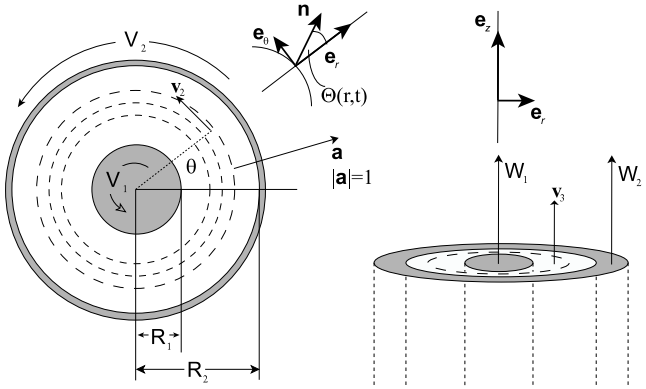

Figure 1. The geometrical set-up for cylindrical Couette flow. The SmA liquid crystal occupies the annular region between two concentric cylinders of radii R_1 and R_2 which have corresponding azimuthal angular velocities V_1 and V_2, respectively. The inner and outer cylinder cylinders are also allowed to move in the axial direction with velocities W_1 and W_2, as indicated. The usual cylindrical polar coordinates (r, θ, z) are used, with the z-axis coincident with the common axis of the cylinders. The velocity **v** is primarily assumed to have components v_2 and v_3 in the $\mathbf{e}_\theta$ and $\mathbf{e}_z$ directions, respectively, as shown. The director makes an angle $\Theta(r, t)$ with respect to the local coordinate axes $\mathbf{e}_r$ and $\mathbf{e}_\theta$, as shown. The smectic layers are denoted by the dashed lines.

for incompressible SmA, namely α_4, τ_1 and τ_2 [1, 2]. External body forces will be neglected. Analogies will be drawn with the anisotropic fluid and nematic cases outlined by Stewart [3] and first published by Atkin and Leslie [12]. However, we expect to see other phenomena associated with the layer structure and layer permeation which are features unique to smectic liquid crystals. The concept of permeation in a liquid crystal itself is unique as the smectic liquid crystal molecules provide the membrane (layers) through which the molecules themselves pass. In the model considered, we shall not impose the condition $\mathbf{a} \equiv \mathbf{n}$ throughout the sample. Instead, we allow the director to rotate, making it possible for the director to make an angle with the radial coordinate direction. In doing so, we are allowing a possible director alignment change to relax the system as the inner and outer cylinders rotate.

In section 2 we employ the dynamic theory for SmA liquid crystals of Stewart [1] to construct (to second order) the relevant governing equations for the system examined. In section 3 the governing equations are solved exactly for a velocity ansatz which includes azimuthal and axial flow to provide forms for the smectic tilt angle, layer displacement function and velocity. Comments are made on the significance of the form of the Lagrange multiplier and the calculation of the pressure. Final conclusions and considerations on future work are given in section 4. Appendix provides a brief summary of the relevant parts of the dynamic theory used to construct the appropriate governing equations.

2. Set-up of the dynamic equations

The dynamic theory for SmA liquid crystals presented by Stewart [1] allows the possibility of the layer normal $\mathbf{a}$ and the director $\mathbf{n}$ not necessarily coinciding in some equilibrium and non-equilibrium situations. As the layer structure could be under considerable stress in this situation, due to the applied boundary conditions at the inner and outer radii, the theory in [1] will be used to model the dynamics of this Couette flow. In order to proceed we must assume forms for the director, the smectic layer function, and the velocity. From figure 1, the director $\mathbf{n}$ can be expressed as

$$\mathbf{n} = \mathbf{e}_r \cos\Theta(r,t) + \mathbf{e}_\theta \sin\Theta(r,t) \tag{2.1}$$

where $\Theta(r,t)$ is a possible (but not necessary) angle between the director and the radial vector. Note that this is still in the SmA phase and we assume $\Theta = 0$ in equilibrium. However, we allow for the possibility of $\Theta \neq 0$ in non-equilibrium situations. Solutions to the resulting governing equations will be sought in which the velocity, expressed in cylindrical coordinates, is of the form

$$\mathbf{v} = \mathbf{e}_\theta r v_2(r,t) + \mathbf{e}_z v_3(r,t) \tag{2.2}$$

where $v_2(r,t)$ and $v_3(r,t)$ are undetermined functions describing the flow in the azimuthal and axial directions, respectively.

We assume here that there is strong homeotropic anchoring of the director on the boundaries, i.e. SmA at the boundaries with the layer normal in the radial direction. Therefore we have the constraints

$$\Theta(R_1,t) = \Theta(R_2,t) = 0. \tag{2.3}$$

We describe the layers via the linear function

$$\Phi(r,t) = r - u(r,t), \tag{2.4}$$

where $u(r,t)$ is the assumed layer displacement away from the equilibrium state. This form for the layer function has been used elsewhere [13–15], although it is possible to use a nonlinear form of the smectic layer function [8, 10, 11]. In this case, we note that in setting $\Phi \equiv r$ for the equilibrium state results in $\mathbf{a} \equiv (1,0,0)$, which we would expect from physical considerations. On displacing the layers we would expect some change in the unit layer normal, especially if the layer displacement is a function of the azimuthal angle θ or height z. We assume here, for simplicity, that the layer displacement and tilt angle are only functions of the radius r and time t. A similar assumption can be seen in the work by Atkin [16]. From equation (2.4) we compute the layer normal, to first order, as

$$\mathbf{a} = \frac{\nabla\Phi}{|\nabla\Phi|} = \mathbf{e}_r. \tag{2.5}$$

We note that the layer displacement term u is absent from the layer normal $\mathbf{a}$. However, we will see that the rôle of u is predominant in the permeation equations. This is to be expected, since the layer direction may remain relatively fixed where permeative flow of material across smectic layers can occur; this is one of the signature features of the smectic phase. We comment that the liquid crystal shall resemble that of classical SmA at equilibrium, only when the director is aligned with the radial coordinate direction, that is, when $\Theta(r,t) \equiv 0$.

The proposed form of the energy density of a smectic A liquid crystal is given by [1]

$$w_{\mathrm{A}} = \tfrac{1}{2}K_1^{\mathrm{n}}(\nabla\cdot\mathbf{n})^2 + \tfrac{1}{2}K_1^{\mathrm{a}}(\nabla\cdot\mathbf{a})^2 + \tfrac{1}{2}B_0(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2)^2 + \tfrac{1}{2}B_1\left(1 - (\mathbf{n}\cdot\mathbf{a})^2\right), \tag{2.6}$$

where K_1^{n} represents the usual elastic splay deformation of the director $\mathbf{n}$ while K_1^{a} is a measure of the bending of the smectic layers; both K_1^{n} and K_1^{a} are positive elastic constants. The term B_0 is the layer compression constant and B_1 is a measure of the strength of the coupling between $\mathbf{n}$ and $\mathbf{a}$. We note that de Gennes [17] has considered higher order approximations to an energy density but excluded the possibility of modelling large separations of $\mathbf{n}$ from the local layer normal. In an alternative nonlinear theory which allows the separation of $\mathbf{n}$ from the classical description of the layer normal, a higher order B_1 term can be included when a non-unit vector approach is adopted [18, 19]. We also note that terms up to second order have been discussed by Nakagawa [20].

The governing equations corresponding to the conservation of linear momentum, conservation of angular momentum and a permeation equation may be derived from those in [1]; the relevant equations are summarized in the appendix. These governing equations can be expressed in physical components

using standard transformations to cylindrical coordinates [3, p 325]. To second order, we have

$$\tilde{p}_{,r} - \rho_0 r v_2^2 + \frac{1}{r} B_0 (\cos\Theta - 1) - B_0 \cos\Theta \left(\frac{1}{r} u_{,r} + u_{,rr} \right) + 2 B_0 u_{,r} u_{,rr} + B_0 \sin\Theta \Theta_{,r} \left(u_{,r} - 1 \right) + \frac{B_0}{r} u_{,r}^2 = 0, \tag{2.7}$$

$$\frac{1}{r} \tilde{p}_{,\theta} + \rho_0 r v_{2,t} - \frac{1}{2} r (\alpha_4 + \tau_2) v_{2,rr} - \frac{3}{2} (\alpha_4 + \tau_2) v_{2,r} + \frac{1}{r} B_0 \sin\Theta \left(u_{,r}^2 - u_{,r} - 1 \right) + \frac{1}{r} \sin\Theta \cos\Theta (B_0 + B_1) \left(1 - u_{,r}^2 \right) = 0, \tag{2.8}$$

$$\tilde{p}_{,z} + \rho_0 v_{3,t} - \frac{1}{2r} (\alpha_4 + \tau_2) v_{3,r} - \frac{1}{2} (\alpha_4 + \tau_2) v_{3,rr} = 0, \tag{2.9}$$

$$-\frac{1}{r^2} K_1^n + B_0 \left(u_{,r} + 1 - \cos\Theta \right) + B_1 \cos\Theta - \lambda \cos\Theta = 0, \tag{2.10}$$

$$\lambda \sin\Theta = 0, \tag{2.11}$$

$$u_{,t} - \frac{\lambda_p B_0}{r} (1 + u_r - \cos\Theta) - \lambda_p B_0 u_{,rr} - \lambda_p B_0 \sin\Theta \Theta_{,r} = 0. \tag{2.12}$$

Here $\tilde{p} = p + w_A$ where p is the pressure, w_A is the energy density, ρ_0 is the density and λ_p is the permeation coefficient. The function λ is a Lagrange multiplier arising from the constraint that **n** is a unit vector.

Equation (2.9) dictates that $\tilde{p} = \tilde{p}(r, \theta, t)$ due to the supposed forms for **n** and **v**. Moreover, for a single-valued solution we must have $\tilde{p}(r, \theta, t) = \tilde{p}(r, \theta + 2\pi, t)$ (cf [3, p 201] and [12]), which forces $\tilde{p} = \tilde{p}(r, t)$ and therefore

$$\tilde{p}_{,\theta} \qquad \text{and} \qquad \tilde{p}_{,z} = 0. \tag{2.13}$$

Equation (2.11) provides us with two possible scenarios: either $\lambda = 0$ or $\sin\Theta = 0$. It can be shown that the $\lambda = 0$ case uncovers contradictions [21] and hence we assume $\Theta = 0$ to second order.

3. Exact solutions

We now proceed to solve the five remaining governing equations (2.7)–(2.10) and (2.12) for the five remaining unknowns $p(r, t)$, $v_2(r, t)$, $v_3(r, t)$, $u(r, t)$ and λ. Inserting $\Theta = 0$ into the permeation equation (2.12) uncovers a partial differential equation for the layer displacement in the form

$$\frac{1}{\lambda_p B_0} u_{,t} = \frac{1}{r} u_{,r} + u_{,rr}. \tag{3.14}$$

We assume the layer displacement can be written

$$u(r, t) = U(r) + \bar{u}(r, t), \tag{3.15}$$

where $U(r)$ is the steady-state solution and $\bar{u}(r, t)$ is a time-dependent perturbation to the system. The steady-state solution of equation (3.14) is easily solved to give

$$U(r) = \mathcal{A} + \mathcal{B} \ln r, \tag{3.16}$$

where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary constants. Considering the substitution

$$\hat{t} = B_0 \lambda_p t, \tag{3.17}$$

and allowing the time-dependent perturbation to the layer displacement to be written in the variables separable form of

$$\bar{u} = \bar{U}(r) e^{-\omega^2 \hat{t}}, \qquad \omega^2 > 0, \tag{3.18}$$

allows us to find the solution [22, p 362]

$$\bar{u} = e^{-\omega^2 \hat{t}} \left(\mathcal{C} J_0(\omega r) + \mathcal{D} Y_0(\omega r) \right), \tag{3.19}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary constants and J_0 and Y_0 are the Bessel functions of the first and second kind of order zero, respectively [23]. Hence, the full general solution for the layer displacement function may be written as

$$u(r, t) = \mathcal{A} + \mathcal{B} \ln r + e^{-\omega^2 B_0 \lambda_p t} \left(\mathcal{C} J_0(\omega r) + \mathcal{D} Y_0(\omega r) \right). \tag{3.20}$$

It is clear from the behaviour of the Bessel functions that layers of large radius have smaller displacements. It is also clear from this solution that any small perturbation to the layer displacement decreases as the radius and/or time increases. We may write the relaxation time (from perturbed state to steady state) of the system, τ_r, as

$$\tau_r = \frac{1}{\omega^2 \lambda_p B_0}. \tag{3.21}$$

However, the magnitude of the exponent ω is unknown. Unless boundary conditions can be stated for the layer displacement, information on ω shall remain unknown. Nevertheless, we see that the time for the layers to settle to their steady-state solution is governed by the permeation constant λ_p and layer compression modulus B_0, as anticipated earlier. Consequently, we remark that the main feature of smectics, compared with nematics, is the presence of permeation and its rôle in the decay process with time to finally settle the layers. The result in (3.21) is completely analogous to that derived for the permeation mode by de Gennes and Prost [2, p 420], in which case $\tau_r = 1/q_z^2 \lambda_p B_0$ where q_z is the wavenumber in the direction of the smectic layer normal **a**; for finite samples, the approximation $q_z \sim \pi/d$, where d is the sample depth in the z-direction, is often used [2, pp 362–72]. For a typical sample depth of $d = 10^{-4}$ m and the estimate [2, p 420] $\lambda_p B_0 \simeq 10^{-9}$ m^2 s^{-1}, this gives rise to the estimate $\tau_r \sim 1$ s. This relaxation time will turn out to be considerably longer than the estimates derived below for the relaxation times of the axial and azimuthal velocity disturbances.

The Lagrange multiplier can now be given explicitly as

$$\lambda = B_1 + B_0 \left(\frac{\mathcal{B}}{r} - e^{-\omega^2 B_0 \lambda_p t} (\mathcal{C} \omega J_1(\omega r) + \mathcal{D} \omega Y_1(\omega r)) \right) - \frac{K_1^n}{r^2}. \tag{3.22}$$

This Lagrange multiplier arises from the constraint $|\mathbf{n}| = 1$. The magnitude of the Lagrange multiplier is an indication of how flexible the liquid crystal molecules are. When the permeation constant λ_p increases, the work required of the multiplier increases. This is reasonable physically since

creation or annihilation of layers is related to the rate of permeation [6], and so the permeation would ultimately affect the director and, in turn, the Lagrange multiplier. Furthermore, we can see that increasing the magnitude of B_1, thereby allowing the layers more opportunity to distort, forces the Lagrange multiplier to increase. This is also physically reasonable since we expect the Lagrange multiplier to have to do more work to keep the system intact if there is more flexibility between the layers and the director. We note that increasing the magnitude of the elastic constant term would decrease the magnitude of the Lagrange multiplier, and hence the work required to keep the constraint $|\mathbf{n}| = 1$ intact. Again, this is physically realistic since a more elastic system would be able to orient more easily to accommodate the $|\mathbf{n}| = 1$ condition. Finally, we consider the effect of the radius on the Lagrange multiplier. Consider the case where $r \to 0$, i.e. when we consider the central region of the liquid crystal sample, it is clear from the behaviour of the Bessel functions ($Y_1(r) \to -\infty$ as $r \to 0$) that this would cause $\lambda \to \infty$. This alerts us to the fact that the dynamic equations are breaking down. One interpretation of this is that the liquid crystal is isotropic near the centre of the sample, or has a disclination core.

We now proceed to solve the remaining equations for $v_2(r, t)$, $v_3(r, t)$ and $\tilde{p}(r, t)$. In accord with the results in (2.13), we set $\tilde{p}_{,z} = 0$. Thus equation (2.9) becomes

$$\frac{2\rho_0}{\alpha_4 + \tau_2} v_{3,t} = \frac{1}{r} v_{3,r} + v_{3,rr}. \tag{3.23}$$

Similar to the method employed for the layer displacement function, we assume the axial velocity v_3 takes the form

$$v_3(r, t) = V(r) + \bar{v}_3(r, t), \tag{3.24}$$

where $V(r)$ is the steady-state solution and $\bar{v}_3(r, t)$ is a time-dependent perturbation to the system. Concentrating, for the moment, on the time-dependent solution and making the change of variable

$$\hat{t} = \frac{\alpha_4 + \tau_2}{2\rho_0} t, \tag{3.25}$$

allows us to consider equation (3.23) in the form

$$-\bar{v}_{3,\hat{t}} + \frac{1}{r}\bar{v}_{3,r} + \bar{v}_{3,rr} = 0. \tag{3.26}$$

Considering a similar form of the velocity to that of the layer undulations

$$\bar{v}_3 = \overline{V}_3(r) \mathrm{e}^{-\nu^2 \hat{t}}, \qquad \nu^2 > 0, \tag{3.27}$$

allows us to solve (3.26) in terms of V_3 to find that [22, pp 361–362]

$$\overline{V}_3(r) = \mathcal{C} J_0(\nu r) + \mathcal{D} Y_0(\nu r), \tag{3.28}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary constants. For small perturbations and assuming non-slip conditions on the boundaries, we require

$$\overline{V}_3(R_1) = \overline{V}_3(R_2) = 0. \tag{3.29}$$

Consequently, the two equations which arise from the solution (3.28) and the boundary condition (3.29) can be written in the form

$$\begin{bmatrix} J_0(\nu R_1) & Y_0(\nu R_1) \\ J_0(\nu R_2) & Y_0(\nu R_2) \end{bmatrix} \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.30}$$

For non-zero solutions in $\mathcal{C}$ and $\mathcal{D}$ we require the determinant of the above 2×2 matrix to be identically zero, that is,

$$J_0(\nu R_1) Y_0(\nu R_2) - Y_0(\nu R_1) J_0(\nu R_2) = 0. \tag{3.31}$$

Let ν_1 be the first positive value of ν that satisfies equation (3.31). Then we can write the solution (3.28) as

$$\overline{V}_3(r) = \frac{\mathcal{C}}{Y_0(\nu_1 R_1)} \left(J_0(\nu_1 r) Y_0(\nu_1 R_1) - J_0(\nu_1 R_1) Y_0(\nu_1 r)\right). \tag{3.32}$$

Since $\mathcal{C}$ is arbitrary, and small because of the linearization process, we can write this solution as

$$\overline{V}_3(r) = \mathcal{E} \left(J_0(\nu_1 r) Y_0(\nu_1 R_1) - J_0(\nu_1 R_1) Y_0(\nu_1 r)\right), \tag{3.33}$$

where $\mathcal{E} = \mathcal{C}/Y_0(\nu_1 R_1)$ is small.

The solution of the steady-state problem, with non-slip conditions on the stationary cylinders, is $V_3 = 0$. However, if we consider cylinders which not only rotate but also move in the axial direction, then we introduce the standard (steady-state) solution which can be found throughout the literature, namely,

$$V(r) = \mathcal{F} + \mathcal{G} \ln r, \tag{3.34}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary constants which can be calculated if the velocities of the moving cylinders are known. Assuming the boundary conditions

$$V(R_1) = V_1, \qquad V(R_2) = V_2, \tag{3.35}$$

implies that

$$V(r) = V_1 + (V_2 - V_1) \frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}. \tag{3.36}$$

Hence, the full solution is given by

$$v_3(r, t) = V_1 + (V_2 - V_1) \frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)} + \mathcal{E}(J_0(\nu_1 r) Y_0(\nu_1 R_1) - J_0(\nu_1 R_1) Y_0(\nu_1 r)) \mathrm{e}^{-\nu_1^2 \hat{t}}. \tag{3.37}$$

Here, $\mathcal{E}$ is the amplitude of the initial perturbation. The response time is given via the factor $\mathrm{e}^{-\nu_1^2 \hat{t}}$, and so, recalling the scaling (3.25), the response time τ_r is given by

$$\tau_\mathrm{r} = \frac{2\rho_0}{(\alpha_4 + \alpha_2)\nu_1^2}. \tag{3.38}$$

Using the suggested material parameters [3]

$$\alpha_4 + \tau_2 = 0.05 \text{ Pa s}, \qquad \rho_0 = 1000 \text{ kg m}^{-3}, \tag{3.39}$$

and the experimental parameters reported by Meiboom [24]

$$\begin{aligned} R_1 &= 9 \times 10^{-3} \text{ m}, & R_2 &= 11 \times 10^{-3} \text{ m}, \\ V_1 &= 10^{-4} \text{ m s}^{-1}, & V_2 &= 3 \times 10^{-4} \text{ m s}^{-1}, \end{aligned} \tag{3.40}$$

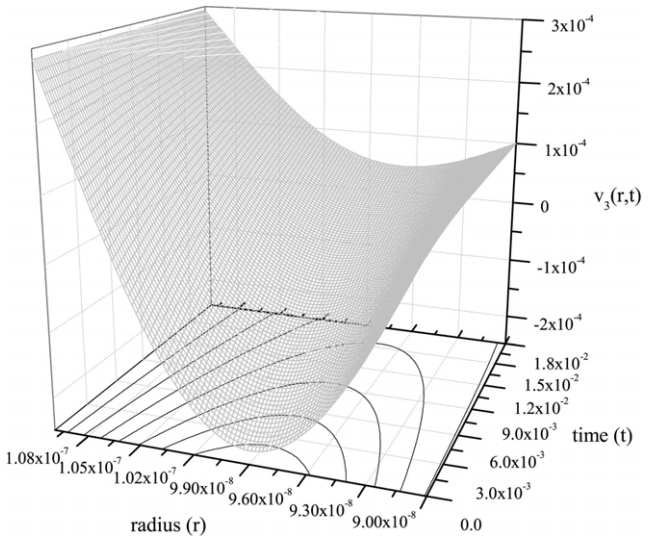

Figure 2. The analytical solution of equation (3.23) for the axial velocity $v_3(r, t)$, given by (3.37), for the material parameters (3.39) and (3.40). For comparison purposes, the arbitrary amplitude $\mathcal{E}$ of the initial perturbation has been set to 10^{-2}.

allows us to calculate the magnitude of the exponent $\nu_1 \approx 1569.9972$ from the relation (3.31). Consequently, the magnitude of the response time is $\tau_r \approx 0.016\,23$ s. This relaxation time is considerably shorter than the estimate derived above for the relaxation time of the smectic layer undulations u, which was of the order of seconds. The resulting axial velocity v_3 given by (3.37) for the parameters (3.40) is plotted in figure 2 where the arbitrary amplitude $\mathcal{E}$ of the perturbation has been set to 10^{-2} for demonstration purposes in order that the perturbation is initially comparable in magnitude to the steady state. We remark here that the response time (3.38) is analogous to that for a Newtonian fluid with constant viscosity η when a shear between parallel plates a distance h apart is applied [25, p 493]: it is given by $\tau_r = \rho_0/\eta\sigma_1^2$ where $\sigma_1 h = \pi$ is the first zero of the first eigenfunction.

The solution for the azimuthal velocity may be found in a similar manner. First, we consider equation (2.8), recalling $\Theta \equiv 0$,

$$\frac{1}{r}\tilde{p}_{,\theta} + \rho_0 r v_{2,t} - \frac{1}{2} r\,(\alpha_4 + \tau_2)\, v_{2,rr} - \frac{3}{2}\,(\alpha_4 + \tau_2)\, v_{2,r} = 0. \tag{3.41}$$

Given that $\tilde{p}_{,\theta} = 0$ (by (2.13)), equation (3.41) may be written as

$$\frac{2\rho_0}{\alpha_4 + \tau_2} v_{2,t} = \frac{3}{r} v_{2,r} + v_{2,rr}. \tag{3.42}$$

We consider the azimuthal velocity to have a steady-state component and a time-dependent perturbation, similar to the axial velocity, i.e.,

$$v_2(r, t) = W(r) + \bar{v}_2(r, t). \tag{3.43}$$

Implementing the same scaling in time described in (3.25) and assuming the time-dependent part of the solution takes the form

$$\bar{v}_2(r, t) = \overline{W}(r) \mathrm{e}^{-\mu^2 \hat{t}}, \qquad \mu^2 > 0, \tag{3.44}$$

allows us to determine the explicit solution [22, pp 361–362]

$$\overline{W}(r) = \frac{\mathcal{H}}{r} J_1(\mu r) + \frac{\mathcal{I}}{r} Y_1(\mu r), \tag{3.45}$$

where $\mathcal{H}$ and $\mathcal{I}$ are arbitrary constants. For non-slip on the boundaries we require the perturbation to vanish there for all times. We therefore require

$$\begin{bmatrix} \frac{J_1(\mu R_1)}{R_1} & \frac{Y_1(\mu R_1)}{R_1} \\ \frac{J_1(\mu R_2)}{R_2} & \frac{Y_1(\mu R_2)}{R_2} \end{bmatrix} \begin{bmatrix} \mathcal{H} \\ \mathcal{I} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.46}$$

Similar to the previous case for v_3 above, for non-zero solutions in the constants $\mathcal{H}$ and $\mathcal{I}$ we require

$$J_1(\mu R_1) Y_1(\mu R_2) - Y_1(\mu R_1) J_1(\mu R_1) = 0. \tag{3.47}$$

Setting μ_1 as the least positive value of μ that satisfies (3.47), and recalling the scaling in time (3.25), allows us to derive the solution as

$$\bar{v}_2(r, t) = \frac{\mathcal{J}}{r}\,(J_1(\mu r) Y_1(\mu_1 R_1) - Y_1(\mu r) J_1(\mu_1 R_1)) \times \mathrm{e}^{-\frac{(\alpha_4+\tau_2)\mu_1^2}{2\rho_0} t}, \tag{3.48}$$

where $\mathcal{J} = \mathcal{H}/Y_1(\mu_1 R_1)$ has been set as the arbitrary initial magnitude of the perturbation. Similar to the previous case, the relaxation rate here is given by

$$\tau_r = \frac{2\rho_0}{(\alpha_4 + \alpha_2)\mu_1^2}. \tag{3.49}$$

The steady-state solution for the azimuthal velocity is given by

$$W(r) = \mathcal{K} + \frac{\mathcal{L}}{r^2}. \tag{3.50}$$

Applying the non-slip conditions

$$W(R_1) = W_1, \qquad W(R_2) = W_2, \tag{3.51}$$

allows us to obtain the solution for $W(r)$ as

$$W(r) = \frac{R_1^2 W_1 - R_2^2 W_2}{R_1^2 - R_2^2} + \frac{1}{r^2}\frac{(W_2 - W_1) R_1^2 R_2^2}{R_1^2 - R_2^2}. \tag{3.52}$$

Hence, the full general solution of equation (3.43) for the axial velocity v_2 is given by

$$v_2(r, t) = \frac{R_1^2 W_1 - R_2^2 W_2}{R_1^2 - R_2^2} + \frac{1}{r^2}\frac{(W_2 - W_1) R_1^2 R_2^2}{R_1^2 - R_2^2} + \frac{\mathcal{I}}{r}(J_1(\mu r) Y_1(\mu_1 R_1) - Y_1(\mu r) J_1(\mu_1 R_1)) \mathrm{e}^{-\frac{(\alpha_4+\tau_2)\mu_1^2}{2\rho_0} t}. \tag{3.53}$$

The relaxation time of the azimuthal velocity can be calculated once the magnitude of μ_1 is known. Using the experimental data quoted in (3.40) (and setting $W_1 = V_1$, $W_2 = V_2$) allows us to numerically calculate $\mu_1 \approx 1573.191$ s. Consequently, using the parameter values detailed in (3.39) allows us to find a relaxation time of $\tau_r \approx 0.016\,16$ s. We note that although this is a slightly longer time than that for the axial velocity, it is also considerably shorter than the estimate for the relaxation time of

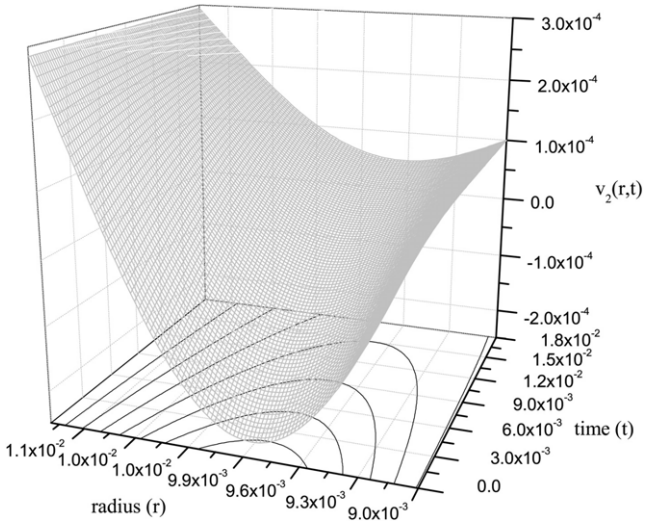

Figure 3. The analytical solution of equation (3.42) for the azimuthal velocity $v_2(r, t)$, given by (3.53), for the material parameters (3.39) and (3.40) with $W_1 = V_1$ and $W_2 = V_2$. For plotting purposes, the arbitrary amplitude $\mathcal{I}$ of the initial perturbation has been set to 10^{-4}.

the smectic layer undulations, as was noted in the case for the axial velocity. The azimuthal velocity is plotted in figure 3. Here, the arbitrary constant has been set as $\mathcal{I} = 10^{-4}$ for comparison with the steady state.

The pressure p can be determined exactly via the function $\tilde{p}(r, t)$ by a direct integration of equation (2.7) using the solutions obtained for the layer undulation function and the two velocity functions. However, due to the complicated nature of the integrals involved, and because it is not needed to determine the results derived above, we do not compute the pressure here.

4. Conclusions and further work

The complete solution to the Couette flow problem described in section 1 is given by the smectic tilt angle Θ, the layer displacement u, the axial velocity component v_3 and the azimuthal velocity component v_2; these results are given by $\Theta = 0$ and equations (3.20), (3.37) and (3.53), respectively. Plots of the axial and azimuthal velocities are presented in figures 2 and 3, respectively. The Lagrange multiplier λ is given by equation (3.22) and the pressure can be calculated via the linear momentum equation (2.7) and the solutions (3.20), (3.37) and (3.53).

As previously mentioned, the dynamic theory of SmA liquid crystals proposed by Stewart [1] was employed in this work to compute the governing equations of the Couette system. This theory was used as it allowed for the director and layer normal to be non-equivalent. However, in the case studied we find that for a non-zero Lagrange multiplier, we are forced into the constraint $\mathbf{a} \equiv \mathbf{n}$. This result is of physical significance; it shows that the liquid crystal will not accommodate a possible misalignment of the director away from the direction of the SmA layer normal in order to reduce the stress on the layers cause by the rotations of the boundary cylinders. Thus, to the linear order of approximation used above, other phenomena, such as permeation, are driving the relaxation. As $\mathbf{a} \equiv \mathbf{n}$, it might be expected that the full dynamic theory for SmA is no longer necessary: however, the permeation equation plays an important rôle in providing a mechanism that influences the relaxation of the smectic layers, through the permeation constant λ_p. Whether the condition $\mathbf{a} \equiv \mathbf{n}$ persists to higher orders of approximation is presently unknown, but it would appear highly unlikely that $\mathbf{a} \equiv \mathbf{n}$ for the fully nonlinear dynamic equations for Couette flow, given that $\mathbf{a} \not\equiv \mathbf{n}$ for nonlinear static problems [8, 9, 11].

Once it was discovered that the director and layer normal were coincident, the layer displacement function and Lagrange multiplier were computed. The layer displacement function was found to be stable in time with a relaxation time $\tau_r = 1/(\omega^2 \lambda_p B_0)$ for some relaxation parameter ω. As indicated above, this relaxation parameter may be estimated through the approximation $\omega \sim \pi/d$, where d is a typical sample depth, whenever an estimate for $\lambda_p B_0$ is available. The Lagrange multiplier evaluated at equation (3.22) also provides information about the system. As the Lagrange multiplier is a measure of the work required to keep the $|\mathbf{n}| = 1$ constraint intact, we see that as the permeation constant λ_p, layer compression constant B_0 or the coupling strength B_1 increase, then the work required by the multiplier increases. Also, if the elastic contribution K_1 is increased, the magnitude of the Lagrange multiplier decreases. All of these situations are expected physically. The concept of permeation has an important rôle in the dynamics of smectic liquid crystals and has been considered previously [1, 2, 26] and this work further emphasizes its influence.

Neglecting a time-dependence in the velocity profile allows us to arrive at results identical to those originally published by Leslie [27] for anisotropic fluids. Once a time-dependence in the velocity is considered, we can solve the linear momentum equations exactly to find a form for the velocity in both the azimuthal and axial directions. These solutions are dependent on two parameters ν and μ which must satisfy the relations (3.31) and (3.47), respectively. The least values of these parameters that satisfy these relations can be calculated numerically, given that the radii of the cylinders are known. These values lead to estimates for the relaxation times for the axial and azimuthal relaxation times given by, respectively, equations (3.38) and (3.49). For typical material parameters, both these relaxation times were considerable shorter than a basic estimate for the relaxation time of the smectic layer undulations given by (3.21). It would appear that small fluctuations to the flow will dissipate more rapidly than fluctuations to the smectic layer structure. It is hoped that these results will encourage more refined measurements for the magnitude of the combined constant $\lambda_p B_0$, via (3.21), and the viscosity coefficient $\alpha_4 + \tau_2$, via the relaxation rates τ_r in (3.38) and (3.49). Since $\alpha_4 = 2\eta$, where η is the well-known standard isotropic Newtonian viscosity, this result could lead to revised approximations for the magnitude of the viscosity coefficient τ_2 if α_4 is estimated from the literature for nematic liquid crystals. This would provide novel information on SmA viscosities.

Appendix

The appropriate parts of the SmA dynamic theory formulated in [1] will now be summarized. Cartesian tensor notation and the summation convention will be used, where any index that is repeated precisely twice in an expression is summed from 1 to 3. Partial differentiation with respect to the variable x_j is denotes by a subscript j preceded by a comma. For example, $a_{i,j}$ denotes the partial derivative of the ith component of $\mathbf{a}$ with respect to the jth spatial coordinate and $a_{i,i}$ represents the divergence of $\mathbf{a}$. The layer normal $\mathbf{a}$ is given by

$$a_i = \frac{\Phi_{,i}}{|\nabla\Phi|}, \qquad a_i a_i = 1, \tag{A.1}$$

where the smectic layers are modelled by the layer function Φ. The usual Oseen [28] constraint, $\nabla \times \mathbf{a} = \mathbf{0}$ will not be imposed for the dynamics: small distortions to the lamellar-like layer structure of SmA generally violate this constraint. The director must satisfy the constraint

$$n_i n_i = 1. \tag{A.2}$$

The incompressibility condition is given by

$$v_{i,i} = 0, \tag{A.3}$$

where $\mathbf{v}$ is the velocity. The rate of strain tensor A and vorticity tensor W are second order tensors defined in the usual way by

$$A_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad W_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i}), \tag{A.4}$$

and, following the standard procedure for nematics, the co-rotational time flux $\mathbf{N}$ of the director $\mathbf{n}$ is introduced as

$$\mathbf{N} = \dot{\mathbf{n}} - \mathsf{W}\mathbf{n}. \tag{A.5}$$

The equations that arise from the balance law for linear momentum in the absence of any external forces are

$$\rho\dot{v}_i = -\tilde{p}_{,i} + \tilde{g}_j n_{j,i} + |\nabla\Phi| a_i J_{j,j} + \tilde{t}_{ij,j}, \tag{A.6}$$

where ρ is the density, $\tilde{p} = p + w_{\mathrm{A}}$ where p is the pressure and w_{A} is the energy density, and $\mathbf{J}$ is defined by

$$J_i = -\frac{\partial w_{\mathrm{A}}}{\partial\,\Phi_{,i}} + \frac{1}{|\nabla\Phi|}\left[\left(\frac{\partial w_{\mathrm{A}}}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w_{\mathrm{A}}}{\partial a_{\mathrm{p}}}\right](\delta_{pi} - a_{\mathrm{p}} a_i). \tag{A.7}$$

A superposed dot represents the usual material time derivative and we remark that $\mathbf{J}$ is sometimes called a 'phase flux' term. It is a natural nonlinear extension to the versions discussed by Auernhammer *et al* [4, 5], E [26] and de Gennes and Prost [2]. When only α_4, τ_1 and τ_2 are considered as the key viscosity coefficients then the constitutive equations for the viscous stress $\tilde{t}_{ij}$ and dynamic contribution $\tilde{g}_i$ in an incompressible SmA liquid crystal are given by, respectively,

$$\begin{aligned}\tilde{t}_{ij} &= \alpha_1(n_k A_{kp} n_{\mathrm{p}}) n_i n_j + \tau_1(a_k A_{kp} a_{\mathrm{p}}) a_i a_j \\ &\quad + \tau_2(a_i A_{j\mathrm{p}} a_{\mathrm{p}} + a_j A_{i\mathrm{p}} a_{\mathrm{p}}), \qquad \tilde{g}_i = 0.\end{aligned} \tag{A.8}$$

The viscosities α_4, τ_1 and τ_2 are analogous to the usual incompressible SmA viscosities [26, equation (3.33)]. Under these circumstances, the balance of angular momentum in the absence of any external forces, leads to the equations

$$\left(\frac{\partial w_{\mathrm{A}}}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w_{\mathrm{A}}}{\partial n_i} = \lambda n_i, \tag{A.9}$$

where the scalar function λ is a Lagrange multiplier that arises from the constraint (A.2) and can usually be identified from by taking the scalar product of (A.9) with $\mathbf{n}$. The permeation equation is

$$\dot{\Phi} = -\lambda_{\mathrm{p}} J_{i,i}, \tag{A.10}$$

where $\lambda_{\mathrm{p}} \geqslant 0$ is the permeation coefficient which is related the layer flux through a stationary medium [2]. Equations (A.2), (A.3), (A.6), (A.9) and (A.10) provide nine equations in the nine unknowns Φ, n_i, v_i, p and λ; the smectic layer normal $\mathbf{a}$ is determined by (A.1) from the solution for Φ.

References

[1] Stewart I W 2007 *Contin. Mech. Thermodyn.* **18** 343–60
[2] de Gennes P G and Prost J 1993 *The Physics of Liquid Crystals* 2nd edn (Oxford: Oxford University Press)
[3] Stewart I W 2004 *The Static and Dynamic Continuum Theory of Liquid Crystals* (London: Taylor and Francis)
[4] Auernhammer G K, Brand H R and Pleiner H 2000 *Rheol. Acta* **39** 215–22
[5] Auernhammer G K, Brand H R and Pleiner H 2002 *Phys. Rev.* E **66** 061707
[6] Ribotta R and Durand G 1977 *J. Physique* **38** 179–204
[7] Soddemann T, Auernhammer G K, Guo H, Dünweg B and Kremer K 2004 *Eur. Phys. J.* E **13** 141–51
[8] Stewart I W 2007 *J. Phys. A: Math. Theor.* **40** 5297–318
[9] Stewart F and Stewart I W 2007 *Mol. Cryst. Liq. Cryst.* **478** 779–88
[10] Walker A J 2008 *J. Phys. A: Math. Theor.* **41** 385205
[11] De Vita R and Stewart I W 2008 *J. Phys.: Condens. Matter* **20** 335101
[12] Atkin R J and Leslie F M 1970 *Q. J. Mech. Appl. Math.* **23** S3–24
[13] Stewart I W 2003 *Liq. Cryst.* **30** 909–20
[14] Walker A J 2007 *Electron.-Liq. Cryst. Commun.* www.e-lc.org/presentations/Documents/Alan_james_Walker_2007_06_24_16_54_42.pdf
[15] Walker A J and Stewart I W 2007 *Mol. Cryst. Liq. Cryst.* **478** 788–99
[16] Atkin R J 1970 *Arch. Ration. Mech. Anal.* **38** 224–40
[17] de Gennes P G 1972 *Solid State Commun.* **10** 753–6
[18] McKay G and Leslie F M 1997 *Eur. J. Appl. Math.* **8** 273–80
[19] McKay G 2004 *J. Non-Newton. Fluid Mech.* **119** 115–22
[20] Nakagawa M 1990 *Liq. Cryst.* **8** 651–75
[21] Walker A J 2008 Theoretical studies of smectic liquid crystals subjected to flow, perturbations, magnetic fields and various applied boundary conditions *PhD Thesis* University of Strathclyde
[22] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
[23] Sneddon I N 1980 *Special Functions of Mathematical Physics and Chemistry* (London: Longman)
[24] Meiboom S and Hewitt R C 1973 *Phys. Rev. Lett.* **30** 261–3
[25] Tanner R I 2000 *Engineering Rheology* 2nd edn (Oxford: Oxford University Press)
[26] E W 1997 *Arch. Ration. Mech. Anal.* **137** 159–75
[27] Leslie F M 1966 *Q. J. Mech. Appl. Math.* **19** 357–70
[28] Oseen C W 1933 *Trans. Faraday Soc.* **29** 883–99